Optimal N-to-M Cloning of Quantum Coherent States

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(May 2000)

The cloning of continuous quantum variables is analyzed based on the concept of Gaussian cloning machines, i.e., transformations that yield copies that are Gaussian mixtures centered on the state to be copied. The optimality of Gaussian cloning machines that transform N identical input states into M output states is investigated, and bounds on the fidelity of the process are derived via a connection with quantum estimation theory. In particular, the optimal N-to-M cloning fidelity for coherent states is found to be equal to $MN/(MN+M-N)$.

PACS numbers: 03.65.Bz, 03.67.-a, 89.70.+c

Cloning denotes an operation by which the unknown state of a system is copied. For reasons rooted at the linearity of quantum mechanics, it turns out that when the system to be copied is quantum mechanical, cloning cannot be performed exactly [1]. Then, a natural question that arises is "to what extent can the copies resemble the original, in accordance with quantum mechanics?" [2]. This is the problem of optimal quantum cloning, which has now been extensively studied for quantum bits [3–7], and, more generally, for d-level systems [8–10]. The present paper investigates the question of optimal cloning for *continuous* quantum variables. Examples of continuous variables include the position and momentum of a particle, or the two quadratures of a quantized electromagnetic field. We will consider a quantum system described in terms of two canonically conjugate operators with continuous spectra (referred to as $\hat{x}$ and $\hat{p}$, with eigenvalues x and p, respectively). Precisely because they are conjugate, $\hat{x}$ and $\hat{p}$ cannot be both copied exactly. Nevertheless, approximate cloning can be achieved if the copies are not required to be exact. Then, the issue of optimal cloning amounts to find the best tradeoff between position and momentum errors induced by cloning.

In this paper, we shall consider $N \to M$ symmetric Gaussian cloners (SGC), defined as a linear completely positive map $C_{N,M}$ transforming N identical replicas of an unknown quantum state $|\psi\rangle$ belonging to an infinite-dimensional Hilbert space $\mathcal{H}$ into $M \geq N$ imperfect clones. The joint state of these clones $\rho_M = C_{N,M}(|\psi^{\otimes N}\rangle\langle\psi^{\otimes N}|)$ is required to be supported on the symmetric subspace of $\mathcal{H}^{\otimes M}$, and is such that the partial trace over all outputs except one is the bi-variate Gaussian mixture

$$\begin{aligned}\rho_1 &= \mathrm{Tr}_{M-1}(\rho_M)\\ &= \frac{1}{\pi\sigma^2_{N,M}} \int d^2\beta \; e^{-|\beta|^2/\sigma^2_{N,M}} \; D(\beta)|\psi\rangle\langle\psi|D^\dagger(\beta)\end{aligned} \tag{1}$$

where the integral is performed over all values of $\beta = (x+ip)/\sqrt{2}$ in the complex plane ($\hbar = 1$), and the operator $D(\beta) = \exp(\beta\hat{a}^\dagger - \beta^*\hat{a})$ achieves a displacement of x in position and p in momentum, with $\hat{a}$ and $\hat{a}^\dagger$ denoting the destruction and creation operators, respectively [11]. Thus, the copies yielded by a SGC are affected by an equal Gaussian noise $\sigma_x^2 = \sigma_p^2 = \sigma^2_{N,M}$ on the conjugate variables x and p. (It will turn out that the resulting cloning fidelity $f = \langle\psi|\rho_1|\psi\rangle$ is invariant for all coherent states of $\hat{x}$ and $\hat{p}$.) The symmetry of the cloner also obviously implies that the M copies are characterized each by the same density operator ρ_1.

The issue of the duplication ($N=1$, $M=2$) of quantum information carried by a continuous variable has been treated in a previous paper [12], where an explicit Gaussian $1 \to 2$ cloning transformation was proposed. It was shown that the noise variance induced by this cloner is $\sigma^2_{1,2} = 1/2$, so that the resulting cloning fidelity for coherent states is $f_{1,2} = 2/3$. This fidelity is invariant under translations and rotations in phase space, so that this Gaussian cloner can be thought of as the analogue for coherent states of the universal cloning machine for quantum bits [2]. The present work investigates the optimality of this $1 \to 2$ cloner, and extends these considerations to $N \to M$ continuous cloners. More specifically, we address the question of "how close" the output state [Eq. (1)] can be from the input state $|\psi\rangle$. We find that a lower bound on the noise variance $\sigma^2_{N,M}$ is given by

$$\overline{\sigma}^2_{N,M} = \frac{M-N}{MN} \tag{2}$$

implying in turn that the *optimal* $N \to M$ cloning fidelity for coherent states is bounded by

$$f_{N,M} = \frac{MN}{MN+M-N} \tag{3}$$

First, let us demonstrate that the bound (2) is achieved with the $1 \to 2$ SGC derived in [12], so that the latter is optimal for coherent states (optimality was only conjectured in [12]). Our proof is directly connected to the problem of simultaneously measuring a pair of conjugate observables on a single quantum system. It is known

(see e.g. [13]) that any attempt to measure $\hat{x}$ and $\hat{p}$ simultaneously on a quantum system is constrained by the inequality

$$\sigma_x^2(1)\ \sigma_p^2(1) \geq 1 \tag{4}$$

where $\sigma_x^2(N)$ and $\sigma_p^2(N)$ denote the variance of the measured values of $\hat{x}$ and $\hat{p}$, respectively, when N replicas of the state are available. (The case where $N > 1$ will be considered later on.) So, the best possible simultaneous measurement of $\hat{x}$ and $\hat{p}$ with a same precision satisfies $\sigma_x^2(1) = \sigma_p^2(1) = 1$. Compared with the intrinsic noise of a minimum-uncertainty wave packet $\sigma_x^2 = \sigma_p^2 = 1/2$, we see that the joint measurement of x and p effects an additional noise of minimum variance 1/2 [13]. Now, let a coherent state $|\alpha\rangle$ be processed by a $1 \to 2$ SGC, and let $\hat{x}$ be measured at one output of the cloner while $\hat{p}$ is measured at the other output. As cloning should obey inequality (4), we must have

$$\Delta\hat{x}^2\ \Delta\hat{p}^2 \geq 1 \tag{5}$$

where $\Delta\hat{x}^2$ ($\Delta\hat{p}^2$) refers to the usual variance of observable $\hat{x}$ ($\hat{p}$) measured on ρ_1. Using Eq. (1), it gives

$$(\delta\hat{x}^2 + \sigma_{1,2}^2)(\delta\hat{p}^2 + \sigma_{1,2}^2) \geq 1 \tag{6}$$

where $\delta\hat{x}^2$ ($\delta\hat{p}^2$) is the intrinsic variance of $\hat{x}$ ($\hat{p}$) measured on the input state, while $\sigma_{1,2}^2$ is the noise variance induced by the cloner. Now, using the uncertainty principle $\delta\hat{x}^2\delta\hat{p}^2 \geq 1/4$ and the identity $a^2 + b^2 \geq 2\sqrt{a^2b^2}$, we conclude that the noise variance is constrained by

$$\sigma_{1,2}^2 \geq \overline{\sigma}_{1,2}^2 = 1/2 \tag{7}$$

implying that the cloner presented in [12] is optimal.

Let us now consider the general problem of optimal $N \to M$ Gaussian cloning. Our proof is connected to quantum state estimation theory similarly to what was done for quantum bits in [14], the key idea being that cloning should not be a way of circumventing the noise limitation encountered in any measuring process. More specifically, our bound relies on the fact that cascading a $N \to M$ cloner with a $M \to L$ cloner results in a $N \to L$ cloner which cannot be better that the *optimal* $N \to L$ cloner. We make use of the property that cascading two SGCs results in a single SGC whose variance is simply the sum of the variances of the two component SGCs (see Appendix). Hence, the variance $\overline{\sigma}_{N,L}^2$ of the *optimal* $N \to L$ SGC must satisfy $\overline{\sigma}_{N,L}^2 \leq \sigma_{N,M}^2 + \sigma_{M,L}^2$. In particular, if the $M \to L$ cloner is itself optimal and $L \to \infty$,

$$\overline{\sigma}_{N,\infty}^2 \leq \sigma_{N,M}^2 + \overline{\sigma}_{M,\infty}^2 \tag{8}$$

Since the limit of $C_{N,M}$ with $M \to \infty$ corresponds to a measurement [3], Eq. (8) implies that cloning the N replicas of a system before measuring the M resulting clones does not provide a mean to enhance the accuracy of a direct measurement of the N replicas.

Let us now estimate $\overline{\sigma}_{N,\infty}^2$, that is, the variance of an optimal joint measurement of $\hat{x}$ and $\hat{p}$ on N replicas of a system. From quantum estimation theory [15], we know that the variance of the measured values of $\hat{x}$ and $\hat{p}$ on a single system, respectively $\sigma_x^2(1)$ and $\sigma_p^2(1)$, are constrained by

$$g_x\sigma_x^2(1) + g_p\sigma_p^2(1) \geq g_x\delta\hat{x}^2 + g_p\delta\hat{p}^2 + \sqrt{g_xg_p} \tag{9}$$

for all values of the constants $g_x, g_p > 0$. Note that, for each value of g_x and g_p, a specific POVM based on a resolution of identity in terms of squeezed states (whose squeezing parameter r is a function of g_x and g_p) achieves this bound (see [15]). Moreover, when measurement is performed on N independent and identical systems, the r. h. s. of (9) is reduced by a factor N^{-1}, as in classical statistics [16]. So, applying N times the optimal single-system POVM is the best joint measurement when N replicas are available since it yields $\sigma_x^2(N) = N^{-1}\sigma_x^2(1)$ and $\sigma_p^2(N) = N^{-1}\sigma_p^2(1)$. Hence, using Eq. (9) for a coherent state ($\delta\hat{x}^2 = \delta\hat{p}^2 = 1/2$) and requiring $\sigma_x^2(N) = \sigma_p^2(N)$, the tightest bound is obtained for $g_x = g_p$. It yields $\overline{\sigma}_{N,\infty}^2 = 1/N$, which, combined with Eq. (8), gives the minimum noise variance induced by cloning, Eq. (2).

It is now easy to compute the fidelity of the optimal $N \to M$ SGC when a coherent state $|\alpha\rangle$ is copied. Using Eq. (1) and the identity $|\langle\alpha|\alpha'\rangle|^2 = \exp(-|\alpha - \alpha'|^2)$, we obtain

$$f_{N,M} = \langle\alpha|\rho_1|\alpha\rangle = \frac{1}{1 + \overline{\sigma}_{N,M}^2} \tag{10}$$

which results in Eq. (3). As expected, all coherent states are copied with a same fidelity. (Note, however, that this property does not extend to all states of $\mathcal{H}$.) Equations (2) and (3) are consistent with the known result for a $1 \to 2$ continuous cloner, i. e., $\overline{\sigma}_{1,2}^2 = 1/2$ and $f_{1,2} = 2/3$ [12]. In addition, they yield the obvious result $\overline{\sigma}_{N,N}^2 = 0$ and $f_{N,N} = 1$, confirming that the optimal $N \to N$ cloning map is just the identity. Furthermore, they fulfill the natural requirement that the cloning fidelity increases with the number of input replicas. For instance, considering a $kN \to kM$ SGC with a positive integer k, we find that $\frac{\partial\overline{\sigma}_{N,M}^2}{\partial k} < 0$ (and $\frac{\partial f}{\partial k} > 0$). At the limit $N \to \infty$, we have $f \to 1$, $\forall M$, that is, classical copying is allowed. Finally, for $M \to \infty$, that is, for an optimal measurement, we get $f \to N/(N+1)$. In particular, it implies that the best simultaneous measurement of $\hat{x}$ and $\hat{p}$ on a single system gives a fidelity 1/2, a well-known result.

It is worth noting that optimally cloning squeezed states requires a variant of these SGCs, just as in [12]. Let us consider for instance a family of quadrature squeezed states with squeezing parameter r. For such a family, the best symmetric cloner must have the form of Eq. (1), but using the definition $\beta = (\frac{x}{\sigma} + i\sigma p)/\sqrt{2}$ with $\sigma = \exp(r)$.

These cloners naturally generalize the SGCs and gives the same cloning fidelity, Eq. (3), for those squeezed states.

In conclusion, we have established a link between optimality of $N \to M$ symmetric Gaussian cloners and the impossibility of simultaneously measuring two conjugate observables $\hat{x}$ and $\hat{p}$. This results in a lower bound on the noise induced by cloning. The optimal cloning fidelity for coherent states was then derived, and was found to be independent of which coherent state is to be copied. The optimal cloning of squeezed states was also found to be equivalent to that of coherent states, as expected since the former can always be obtained by applying a canonical transformation on the latter. It is unknown whether a cloner specifically devised for other classes of states might yield a fidelity exceeding Eq. (3). However, since minimum-uncertainty states are the closest to classical states, we conjecture that SGCs achieve the *best* possible fidelity if we require the cloner to be covariant under rotations and translations in the phase space. Finally, even though the explicit transformation achieving the $1 \to 2$ optimal SGC is known [12], finding the $N \to M$ cloning transformation that attains the maximum fidelity is still an open question.

Appendix. We now prove that the variances of two cascaded cloners add. Consider a $N \to M$ SGC, followed by a $M \to L$ SGC. Let ρ be an arbitrary density operator supported on $\mathcal{H}^{\otimes \mathcal{M}}$. Since it is self-adjoint and compact, ρ has a denumerable spectrum: it can be expanded as $\rho = \sum_{i=1}^{\infty} \lambda_i |\xi_i\rangle\langle\xi_i|$ with $\langle\xi_i|\xi_j\rangle = \delta_{ij}$, $\lambda_i \geq 0$ and $\sum_{i=1}^{\infty} \lambda_i = 1$. Note that $\forall \epsilon > 0$, $\exists d$ such that $|\sum_{i=1}^{d} \lambda_i - 1| < \epsilon$. Therefore, the output of the first cloner can be decomposed as $\rho_M = \rho_d + \epsilon_d \, B_d$ where $\rho_d = \sum_{i=1}^{d} \lambda_i |\xi_i\rangle\langle\xi_i|$ is supported on a d-dimensional subspace of $\mathcal{H}^{\otimes \mathcal{M}}$, B_d is a bounded operator, and $\lim_{d\to\infty} \epsilon_d = 0$. Since ρ_M belongs to the symmetric subspace of $\mathcal{H}^{\otimes \mathcal{M}}$, so will ρ_d. Hence, we know that we can write ρ_d in the form of a pseudo-mixture of pure product states $\rho_d = \sum_{i=1}^{d} \alpha_i |\phi_i^{\otimes M}\rangle\langle\phi_i^{\otimes M}|$ where the coefficients α_i are not necessarily positive but satisfy $\sum_{i=1}^{d} \alpha_i = 1$ (see [8] or [14]). Thus, when cloning a state $|\psi^{\otimes N}\rangle$, we have

$$C_{N,M}(|\psi^{\otimes N}\rangle\langle\psi^{\otimes N}|) = \sum_{i=1}^{d} \alpha_i |\phi_i^{\otimes M}\rangle\langle\phi_i^{\otimes M}| + \epsilon_d \, B_d \quad (11)$$

Then, since the cloning map $C_{N,M}$ is linear, cascading the two cloners yields $C_{M,L}C_{N,M}(|\psi^{\otimes N}\rangle\langle\psi^{\otimes N}|) = \sum_i \alpha_i C_{M,L}(|\phi_i^{\otimes N}\rangle\langle\phi_i^{\otimes N}|) + \epsilon_d C_{M,L}(B_d)$. As this expression is a density operator (thus bounded) and the first term of its r.h.s. is positive, $C_{ML}(B_d)$ must be bounded. Thus, the second term of the r. h. s. of Eq. (11) becomes negligible when $d \to \infty$. Now, using Eq.(1), we have

$$\begin{aligned} &\mathrm{Tr}_{L-1} C_{M,L} C_{N,M}(|\psi^{\otimes N}\rangle\langle\psi^{\otimes N}|) = \\ &\quad \frac{1}{\pi^2 \sigma_{M,L}^2 \sigma_{N,M}^2} \int d^2\gamma \; d^2\beta \; \mathrm{e}^{-|\gamma|^2/\sigma_{M,L}^2 - |\beta|^2/\sigma_{N,M}^2} \\ &\quad \times D(\gamma+\beta)|\psi\rangle\langle\psi|D^\dagger(\gamma+\beta) + O(\eta_d) \end{aligned} \quad (12)$$

with $\lim_{d\to\infty} \eta_d = 0$. A little algebra then shows that this last expression is a Gaussian mixture centered on the original state whose variance is $\sigma_{M,L}^2 + \sigma_{N,M}^2$.

We are grateful to Serge Massar for very helpful discussions, especially concerning quantum estimation theory. S. I. acknowledges support from the Fondation Universitaire Van Buuren at the Université Libre de Bruxelles.

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